

Off- and on-shell harmonic superspaces for $6D$ SYM theories

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Abstract. It is a brief account of the harmonic superspace formulations of $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ SYM theories in six dimensions. The on-shell $\mathcal{N} = (1,1)$ harmonic superspace is argued to provide an efficient tool of constructing candidate counterterms and other invariants of $\mathcal{N} = (1,1)$ SYM. It allows one, e.g., to find out an essential difference between the single- and double-trace dimension $d=10$ invariants, which could be capable to explain the absence of the three-loop double-trace (non-planar) counterterms in this theory. The defining superspace constraints of $\mathcal{N} = (1,1)$ SYM are solved in terms of $\mathcal{N} = (1,0)$ harmonic superfields.

1. Motivations

During recent years, much attention was paid to the maximally extended (with 16 supercharges) supersymmetric gauge theories in diverse dimensions (see, e.g., [1]),

$$\mathcal{N} = 4, 4D \implies \mathcal{N} = (1,1), 6D \implies \mathcal{N} = (1,0), 10D.$$

The renowned $\mathcal{N} = 4, 4D$ SYM theory was the first example of an UV *finite* theory. Perhaps, it is also *completely integrable* [2]. The $\mathcal{N} = (1,1), 6D$ SYM is not renormalizable by formal counting (the coupling constant is dimensionful) but it is also expected to feature unique properties. In particular, in the perturbation theory it exhibits “dual conformal symmetry”, like its $4D$ counterpart which respects “dual superconformal symmetry” [3]. It can be viewed as an effective theory for description of some particular low energy sectors of string theory, such as D5-brane dynamics. The quantum effective action of D5-brane as a generalization of the microscopic $\mathcal{N} = (1,1)$ SYM action was conjectured to be of non-abelian Born-Infeld type [4, 5]. The $\mathcal{N} = (1,1)$ SYM is anomaly free, as opposed to $\mathcal{N} = (1,0)$ SYM theory.

The $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$ SYM theories provide an appropriate “laboratory” for studying $\mathcal{N} = 8$ supergravity and its some lower \mathcal{N} descendants, which are also non-renormalizable according by the standard counting.

The latest perturbative calculations in $\mathcal{N} = (1,1)$ SYM showed plenty of unexpected cancelations of the UV divergencies. The theory is UV finite up to 2 loops, while at 3 loops only a single-trace (planar) counterterm of canonical dim 10 is needed. The permissible double-trace (non-planar) counterterms do not appear [6] - [8]. Various arguments to explain this

phenomenon were suggested [9] - [12], though the complete understanding of it is still lacking. One could expect the existence of some new non-renormalization theorems in this connection.

The maximal off-shell supersymmetry that one can achieve in $6D$ is $\mathcal{N} = (1, 0)$ supersymmetry. The natural off-shell formulation of $\mathcal{N} = (1, 0)$ SYM theory is given in harmonic $\mathcal{N} = (1, 0), 6D$ superspace [13, 14], generalizing the harmonic $\mathcal{N} = 2, 4D$ one [15, 16]. The harmonic $6D$ formulations were further worked out in [17] - [20] and [21]. The $\mathcal{N} = (1, 1)$ SYM theory in the harmonic formalism is a hybrid of two $\mathcal{N} = (1, 0)$ theories, $[\mathcal{N} = (1, 1) \text{ SYM}] = [\mathcal{N} = (1, 0) \text{ SYM}] + [6D \text{ hypermultiplets}]$, with the second hidden $\mathcal{N} = (0, 1)$ supersymmetry. The natural question is as to how to construct higher-dimension $\mathcal{N} = (1, 1)$ invariants in the $\mathcal{N} = (1, 0)$ superfield approach.

One way to approach this issue is the “brute-force” method. One starts with the appropriate dimension $\mathcal{N} = (1, 0)$ SYM invariant and then completes it to $\mathcal{N} = (1, 1)$ invariant step by step, adding the proper hypermultiplet terms. This approach is rather cumbersome.

Some simplifications arise due to the fact that for finding superfield counterterms it suffices to stay on the mass shell. In a recent paper [21] there was suggested a new approach to constructing higher-dimension $\mathcal{N} = (1, 1)$ invariants. It makes use of the concept of the *on-shell* $\mathcal{N} = (1, 1)$ harmonic superspace with the double set of the harmonic variables $u_i^\pm, u_A^\pm, i = 1, 2; A = 1, 2$ [22]. The novel point of the consideration in [21] is solving the $\mathcal{N} = (1, 1)$ SYM constraints [23, 24] in terms of $\mathcal{N} = (1, 0)$ superfields. The dimension $\underline{d} = 8$ and $\underline{d} = 10$ invariants were constructed in a simple way and an essential difference between the single- and double-trace dimension $\underline{d} = 10$ invariants was observed. The present contribution is a brief survey of the $6D$ harmonic methods, with the main emphasis on their recent applications in [21].

2. $6D$ superspaces and superfields

2.1. $6D$ superspaces

- The customary $\mathcal{N} = (1, 0), 6D$ superspace is parametrized by the following coordinate set:

$$z = (x^M, \theta_i^a), \quad M = 0, \dots, 5, \quad a = 1, \dots, 4, \quad i = 1, 2, \quad (1)$$

- The harmonic $\mathcal{N} = (1, 0)$ superspace amounts to adding $SU(2)$ harmonics to (1):

$$Z := (z, u) = (x^M, \theta_i^a, u^{\pm i}), \quad u_i^- = (u_i^+)^*, \quad u^{+i}u_i^- = 1, \quad u^{\pm i} \in SU(2)_R/U(1). \quad (2)$$

- The *analytic* $\mathcal{N} = (1, 0)$ superspace forms an invariant subspace in (2):

$$\zeta := (x_{(\text{an})}^M, \theta^{+a}, u^{\pm i}) \subset Z, \quad x_{(\text{an})}^M = x^M + \frac{i}{2} \theta_k^a \gamma_{ab}^M \theta_l^b u^{+k} u^{-l}, \quad \theta^{\pm a} = \theta_i^a u^{\pm i}. \quad (3)$$

The differential operators in the analytic basis $Z_A := (x_{(\text{an})}^M, \theta^{+a}, u^{\pm i}, \theta^{-a})$ are defined according to

$$\begin{aligned} D_a^+ &= \partial_{-a}, \quad D_a^- = -\partial_{+a} - 2i\theta^{-b}\partial_{ab}, \quad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\partial_{+a} - \theta^{-a}\partial_{-a} \\ D^{++} &= \partial^{++} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\partial_{-a}, \quad D^{--} = \partial^{--} + i\theta^{-a}\theta^{-b}\partial_{ab} + \theta^{-a}\partial_{+a}, \\ \partial_{\pm a}\theta^{\pm b} &= \delta_a^b, \quad \partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}, \quad \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}. \end{aligned} \quad (4)$$

2.2. $6D$ superfields

The fundamental quantity of $\mathcal{N} = (1, 0)$ SYM theory is the analytic gauge connection $V^{++}(\zeta)$

$$\nabla^{++} = D^{++} + V^{++}, \quad \delta V^{++} = -\nabla^{++}\Lambda, \quad \Lambda = \Lambda(\zeta). \quad (5)$$

The second harmonic (non-analytic) connection $V^{--}(Z)$ entering the covariantized harmonic derivative ∇^{--} ,

$$\nabla^{--} = D^{--} + V^{--}, \quad \delta V^{--} = -\nabla^{--}\Lambda,$$

is expressed through V^{++} from the harmonic “flatness” condition

$$\begin{aligned} [\nabla^{++}, \nabla^{--}] &= D^0 \Leftrightarrow D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0 \\ \Rightarrow V^{--} &= V^{--}(V^{++}, u^\pm). \end{aligned} \quad (6)$$

The off-shell contents of $\mathcal{N} = (1, 0)$ SYM theory is revealed in the Wess-Zumino gauge for the analytic gauge potential:

$$V^{++} = \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^+)_a^3\lambda^{ai}u_i^- - 3(\theta^+)^4\mathcal{D}^{ik}u_i^-u_k^-. \quad (7)$$

Here A_{ab} is the gauge field, λ^{ai} is the gaugino and $\mathcal{D}^{ik} = \mathcal{D}^{ki}$ are the auxiliary fields.

The $\mathcal{N} = (1, 0)$ SYM covariant derivatives are defined as

$$\begin{aligned} \nabla_a^- &= [\nabla^{--}, D_a^+] = D_a^- + \mathcal{A}_a^-, \quad \nabla_{ab} = \frac{1}{2i}[D_a^+, \nabla_b^-] = \partial_{ab} + \mathcal{A}_{ab}, \\ \mathcal{A}_a^-(V) &= -D_a^+V^{--}, \quad \mathcal{A}_{ab}(V) = \frac{i}{2}D_a^+D_b^+V^{--}, \\ [\nabla^{++}, \nabla_a^-] &= D_a^+, \quad [\nabla^{++}, D_a^+] = [\nabla^{--}, \nabla_a^-] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0. \end{aligned} \quad (8)$$

The covariant superfield strengths are constructed by the appropriate connections

$$\begin{aligned} [D_a^+, \nabla_{bc}] &= \frac{i}{2}\varepsilon_{abcd}W^{+d}, \quad [\nabla_a^-, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd}W^{-d}, \\ W^{+a} &= -\frac{1}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+V^{--}, \quad W^{-a} := \nabla^{--}W^{+a}, \\ \nabla^{++}W^{+a} &= \nabla^{--}W^{-a} = 0, \quad \nabla^{++}W^{-a} = W^{+a}, \\ D_b^+W^{+a} &= \delta_b^aF^{++}, \quad F^{++} = \frac{1}{4}D_a^+W^{+a} = (D^+)^4V^{--}, \\ \nabla^{++}F^{++} &= 0, \quad D_a^+F^{++} = 0. \end{aligned} \quad (9)$$

The hypermultiplet is described off shell by an analytic superfield $q^{+A}(\zeta)$, $(A = 1, 2)$, :

$$q^{+A}(\zeta) = q^{+A}(x)u_i^+ - \theta^{+a}\psi_a^A(x) + \text{An infinite tail of auxiliary fields}. \quad (10)$$

2.3. $\mathcal{N} = (1, 0)$ superfield actions

The $\mathcal{N} = (1, 0)$ SYM action was constructed by Boris Zupnik [14]:

$$\begin{aligned} S^{SYM} &= \frac{1}{f^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} \int d^6x d^8\theta du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+u_2^+) \dots (u_n^+u_1^+)}, \\ \delta S^{SYM} &= 0 \Rightarrow F^{++} = 0. \end{aligned} \quad (11)$$

Here, $(u_1^+u_2^+)^{-1}, \dots (u_n^+u_1^+)^{-1}$ are harmonic distributions [16].

The hypermultiplet action, with q^{+A} in adjoint representation of the gauge group, is written down as

$$\begin{aligned} S^q &= -\frac{1}{2f^2} \text{Tr} \int d\zeta^{-4} q^{+A} \nabla^{++} q_A^+, \quad \nabla^{++} q_A^+ = D^{++} q_A^+ + [V^{++}, q_A^+], \\ \delta S^q &= 0 \Rightarrow \nabla^{++} q^{+A} = 0. \end{aligned} \quad (12)$$

The $\mathcal{N} = (1, 0)$ superfield form of the $\mathcal{N} = (1, 1)$ SYM action is a sum of the two superfield actions given above:

$$\begin{aligned} S^{(V+q)} &= S^{SYM} + S^q = \frac{1}{f^2} \left(\int dZ \mathcal{L}^{SYM} - \frac{1}{2} \text{Tr} \int d\zeta^{-4} q^{+A} \nabla^{++} q_A^+ \right), \\ \delta S^{(V+q)} = 0 &\Rightarrow F^{++} + \frac{1}{2} [q^{+A}, q_A^+] = 0, \quad \nabla^{++} q^{+A} = 0. \end{aligned} \quad (13)$$

It is invariant under the second hidden $\mathcal{N} = (0, 1)$ supersymmetry acting as:

$$\delta V^{++} = \epsilon^{+A} q_A^+, \quad \delta q^{+A} = -(D^+)^4 (\epsilon_A^- V^{--}), \quad \epsilon_A^\pm = \epsilon_{aA} \theta^{\pm a}. \quad (14)$$

These transformations have the correct closure among themselves and with the manifest $\mathcal{N} = (1, 0)$ supersymmetry only on shell.

3. Higher-dimensional invariants

3.1. Dimension $\underline{d} = 6$

In the pure $\mathcal{N} = (1, 0)$ SYM theory the $\underline{d} = 6$ invariant is defined uniquely [17]:

$$S_{SYM}^{(6)} = \frac{1}{2g^2} \text{Tr} \int d\zeta^{-4} du \left(F^{++} \right)^2 \sim \text{Tr} \int d^6 x [(\nabla^M F_{ML})^2 + \dots]. \quad (1)$$

It vanishes on shell, when $F^{++} = 0$. Using the results of [18], its $\mathcal{N} = (1, 1)$ completion is defined up to a real parameter

$$\mathcal{L}^{d=6} = \frac{1}{2g^2} \text{Tr} \int du d\zeta^{-4} \left(F^{++} + \frac{1}{2} [q^{+A}, q_A^+] \right) \left(F^{++} + 2\beta [q^{+A}, q_A^+] \right). \quad (2)$$

But it vanishes by itself on the full $\mathcal{N} = (1, 1)$ SYM mass shell! This proves the *one-loop finiteness* of $\mathcal{N} = (1, 1)$ SYM theory.

3.2. Dimension $\underline{d} = 8$

All superfield operators of the canonical dimension $\underline{d} = 8$ in the $\mathcal{N} = (1, 0)$ SYM theory vanish on shell, in accord with the statement of ref. [24]. Can this conclusion be changed upon adding the hypermultiplet terms? We have checked that there exist no $\mathcal{N} = (1, 0)$ off-shell invariants of the dimension $\underline{d} = 8$ which would respect the on-shell $\mathcal{N} = (1, 1)$ invariance.

Surprisingly, the $\underline{d} = 8$ superfield expression which is non-vanishing on shell and respects the on-shell $\mathcal{N} = (1, 1)$ supersymmetry can be constructed by *giving up* the demand of *off-shell* $\mathcal{N} = (1, 0)$ supersymmetry.

An example of such an invariant in $\mathcal{N} = (1, 0)$ SYM theory is very simple

$$\tilde{S}_1^{(8)} \sim \text{Tr} \int d\zeta^{-4} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d}. \quad (3)$$

Indeed, $D_a^+ W^{+b} = \delta_a^b F^{++}$, which vanishes on shell, with $F^{++} = 0$. Thus, W^{+a} is on-shell analytic, for which reason the above action respects $\mathcal{N} = (1, 0)$ supersymmetry on shell. Also, an analogous double-trace on-shell invariant exists.

These invariants possess $\mathcal{N} = (1, 1)$ completions. For (3) such a completion reads

$$\begin{aligned} \mathcal{L}_{(1,1)}^{+4} &= \text{Tr}_{(S)} \left\{ \frac{1}{4} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d} + 3i q^{+A} \nabla_{ab} q_A^+ W^{+a} W^{+b} \right. \\ &\quad - q^{+A} \nabla_{ab} q_A^+ q^{+B} \nabla^{ab} q_B^+ - W^{+a} [D_a^+ q_A^-, q_B^+] q^{+A} q^{+B} \\ &\quad \left. - \frac{1}{2} [q^{+C}, q_C^+] [q_A^-, q_B^+] q^{+A} q^{+B} \right\}. \end{aligned} \quad (4)$$

Here, $\text{Tr}_{(S)}$ stands for the *symmetrized* trace. This Lagrangian is analytic, $D_a^+ \mathcal{L}_{(1,1)}^{+4} = 0$, on the total mass shell $F^{++} + \frac{1}{2}[q^{+A}, q_A^+] = 0$, $\nabla^{++} q^{+A} = 0$, hence it is on-shell $\mathcal{N} = (1, 1)$ supersymmetric.

Though the nontrivial on-shell $\underline{d=8}$ invariants exist, the perturbative expansion for the amplitudes in the $\mathcal{N} = (1, 1)$ SYM theory does not involve divergences at the two-loop level. The reason is that these $\underline{d=8}$ invariants do *not* possess the full off-shell $\mathcal{N} = (1, 0)$ supersymmetry which the physically relevant counterterms should obey.

4. $\mathcal{N} = (1, 1)$ on-shell harmonic superspace

Apart from the fact that the $\underline{d=8}$ terms mentioned above cannot come out as counterterms in $\mathcal{N} = (1, 1)$ SYM theory, they can arise, e.g., as quantum corrections to the effective Wilsonian action. For the pure $\mathcal{N} = (1, 0)$ SYM theory this was recently observed in [20]. It was desirable to have some systematic way of constructing such higher-order $\mathcal{N} = (1, 1)$ invariants. This proves to be possible within the on-shell harmonic $\mathcal{N} = (1, 1)$ superspace.

Let us start by extending the $\mathcal{N} = (1, 0)$ superspace to the $\mathcal{N} = (1, 1)$ one,

$$z = (x^{ab}, \theta_i^a) \Rightarrow \hat{z} = (x^{ab}, \theta_i^a, \hat{\theta}_a^A). \quad (1)$$

Then we introduce the gauge-covariantized spinor derivatives,

$$\nabla_a^i = \frac{\partial}{\partial \theta_i^a} - i\theta^{bi} \partial_{ab} + \mathcal{A}_a^i, \quad \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}_b^A \partial^{ab} + \hat{\mathcal{A}}^{aA}. \quad (2)$$

The superspace constraints defining the $\mathcal{N} = (1, 1)$ SYM theory can be then written as follows [23, 24]:

$$\begin{aligned} \{\nabla_a^{(i}, \nabla_b^{j)}\} &= \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0, \quad \{\nabla_a^i, \hat{\nabla}^{bA}\} = \delta_a^b \phi^{iA} \\ \Rightarrow \nabla_a^{(i} \phi^{j)A} &= \hat{\nabla}^{a(A} \phi^{B)i} = 0 \quad (\text{By Bianchis}). \end{aligned} \quad (3)$$

As the next step, we introduce the $\mathcal{N} = (1, 1)$ harmonic superspace [22],

$$Z = (x^{ab}, \theta_i^a, u_k^\pm) \Rightarrow \hat{Z} = (x^{ab}, \theta_i^a, \hat{\theta}_b^A, u_k^\pm, u_A^\pm), \quad (4)$$

pass to the analytic basis in it and choose the “hatted” spinor derivatives short, $\nabla^{\hat{+}a} = D^{\hat{+}a} = \frac{\partial}{\partial \theta_a^-}$. The set of constraints (3) is equivalently rewritten as

$$\begin{aligned} \{\nabla_a^+, \nabla_b^+\} &= 0, \quad \{D^{\hat{+}a}, D^{\hat{+}b}\} = 0, \quad \{\nabla_a^+, D^{\hat{+}b}\} = \delta_a^b \phi^{\hat{+}+}, \\ [\nabla^{\hat{+}\hat{+}}, \nabla_a^+] &= 0, \quad [\tilde{\nabla}^{++}, \nabla_a^+] = 0, \quad [\nabla^{\hat{+}\hat{+}}, D^{\hat{+}a}] = 0, \quad [\tilde{\nabla}^{++}, D^{\hat{+}a}] = 0, \\ [\tilde{\nabla}^{++}, \nabla^{\hat{+}\hat{+}}] &= 0, \end{aligned} \quad (5)$$

$$\begin{aligned} \nabla_a^+ &= D_a^+ + \mathcal{A}_a^+(\hat{Z}), \quad \tilde{\nabla}^{++} = D^{++} + \tilde{V}^{++}(\hat{\zeta}), \quad \nabla^{\hat{+}\hat{+}} = D^{\hat{+}\hat{+}} + V^{\hat{+}\hat{+}}(\hat{\zeta}), \\ \hat{\zeta} &= (x_{\text{an}}^{ab}, \theta^{\pm a}, \theta_c^\pm, u_i^\pm, u_A^\pm). \end{aligned} \quad (6)$$

5. Solving $\mathcal{N} = (1, 1)$ SYM constraints through $\mathcal{N} = (1, 0)$ superfields

The starting point of our analysis in [21] was the WZ gauge for the extra connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$

$$V^{\hat{+}\hat{+}} = i\theta_a^\pm \theta_b^\pm \hat{\mathcal{A}}^{ab} + \varepsilon^{abcd} \theta_a^\pm \theta_b^\pm \theta_c^\pm \varphi_d^A u_A^\pm + \varepsilon^{abcd} \theta_a^\pm \theta_b^\pm \theta_c^\pm \theta_d^\pm \mathcal{D}^{AB} u_A^\pm u_B^\pm, \quad (1)$$

where $\hat{\mathcal{A}}^{ab}$, φ_d^A and $\mathcal{D}^{(AB)}$ are some $\mathcal{N} = (1, 0)$ harmonic superfields, still arbitrary at this step.

Then the above constraints are reduced to some sets of harmonic equations. We have solved them and, as the eventual result, found that the first harmonic connection V^{++} coincides with the previous $\mathcal{N} = (1, 0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N} = (1, 1)$ objects on the “hatted” variables is fixed as

$$\begin{aligned} V^{\hat{+}\hat{+}} &= i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\mathcal{A}^{ab} - \frac{1}{3}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}D_d^+q^{-\hat{-}} + \frac{1}{8}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, q^{-\hat{-}}] \\ \phi^{+\hat{+}} &= q^{+\hat{+}} - \theta_a^{\hat{+}}W^{+a} - i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}[D_d^+q^{-\hat{-}}, q^{+\hat{-}}] \\ &+ \frac{1}{24}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]]. \end{aligned} \quad (2)$$

Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u_A^{\hat{\pm}}$, $q^{-\hat{\pm}} = q^{-A}(\zeta)u_A^{\hat{\pm}}$ and $W^{+a}, q^{\pm A}$ are just the $\mathcal{N} = (1, 0)$ superfields we dealt with previously. In the course of solving the constraints, there naturally appear the superfield equations of motion

$$\nabla^{++}q^{+A} = 0, \quad F^{++} = \frac{1}{4}D_a^+W^{+a} = -\frac{1}{2}[q^{+A}, q_A^+]. \quad (3)$$

Also, the structure of the spinor covariant derivatives is completely fixed

$$\begin{aligned} \nabla_a^+ &= D_a^+ - \theta_a^{\hat{+}}q^{+\hat{-}} + \theta_a^{\hat{-}}\phi^{+\hat{+}}, \\ \nabla_a^- &= D_a^- - D_a^+V^{--} - \theta_a^{\hat{+}}q^{-\hat{-}} + \theta_a^{\hat{-}}\phi^{-\hat{+}}, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}. \end{aligned} \quad (4)$$

The crucial point of our analysis was the requirement that the vector $6D$ connections in the sectors of hatted and unhatted variables are identical to each other.

The basic advantage of using the constrained $\mathcal{N} = (1, 1)$ strengths $\phi^{\pm\hat{+}}$ for constructing invariants is the very simple transformation rules of $\phi^{\pm\hat{+}}$ under the hidden $\mathcal{N} = (0, 1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}} = -\epsilon_a^{\hat{+}}\frac{\partial}{\partial\theta_a^{\hat{+}}}\phi^{\pm\hat{+}} - 2i\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\partial^{ab}\phi^{\pm\hat{+}} - [\Lambda^{(comp)}, \phi^{\pm\hat{+}}], \quad (5)$$

where $\Lambda^{(comp)}$ is some composite gauge parameter which makes no contribution under the trace.

6. Invariants in $\mathcal{N} = (1, 1)$ superspace

The single-trace dimension $d = 8$ invariant (4) can be readily rewritten in $\mathcal{N} = (1, 1)$ superspace

$$S_{(1,1)} = \int d\zeta^{-4}\mathcal{L}_{(1,1)}^{+4}, \quad \mathcal{L}_{(1,1)}^{+4} = -\text{Tr} \frac{1}{4} \int d\hat{\zeta}^{-4}d\hat{u} (\phi^{+\hat{+}})^4, \quad d\hat{\zeta}^{-4} \sim (D^{\hat{-}})^4 \quad (1)$$

$$\delta\mathcal{L}_{(1,1)}^{+4} = -2i\partial^{ab}\text{Tr} \int d\hat{\zeta}^{-4}d\hat{u} \left[\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\frac{1}{4}(\phi^{+\hat{+}})^4 \right].$$

Analogously, the double-trace $d = 8$ invariant is given by

$$\hat{\mathcal{L}}_{(1,1)}^{+4} = -\frac{1}{4} \int d\hat{\zeta}^{-4}d\hat{u} \text{Tr} (\phi^{+\hat{+}})^2 \text{Tr} (\phi^{+\hat{+}})^2. \quad (2)$$

Now it is easy to construct the single- and double-trace $d = 10$ invariants

$$\begin{aligned} S_1^{(10)} &= \text{Tr} \int dZ d\hat{\zeta}^{-4}d\hat{u} (\phi^{+\hat{+}})^2 (\phi^{-\hat{+}})^2, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}, \\ S_2^{(10)} &= - \int dZ d\hat{\zeta}^{-4}d\hat{u} \text{Tr} (\phi^{+\hat{+}}\phi^{-\hat{+}}) \text{Tr} (\phi^{+\hat{+}}\phi^{-\hat{+}}). \end{aligned} \quad (3)$$

It is notable that the single-trace $\underline{d} = 10$ invariant admits a representation as an integral over the *full* $\mathcal{N} = (1, 1)$ superspace

$$S_1^{(10)} \sim \text{Tr} \int dZ d\hat{Z} d\hat{u} \, \phi^{+\hat{+}} \phi^{-\hat{-}}, \quad \phi^{-\hat{-}} = \nabla^{\hat{-}\hat{-}} \phi^{-\hat{+}}, \quad (4)$$

with $d\hat{Z} \sim (D^{\hat{-}})^4 (D^{\hat{+}})^4$. On the other hand, the double-trace $\underline{d} = 10$ invariant *cannot* be written as the total integral and so looks as being UV *protected*.

This could explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1, 1)$ SYM theory single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed [6], [7], [8]. However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1, 0)$ supergraphs should give invariants in the *off-shell* $\mathcal{N} = (1, 0)$ superspace, not in the *on-shell* $\mathcal{N} = (1, 1)$ superspace. So the above property seems not enough to explain the absence of the double-trace divergences and some additional piece of reasoning is needed.

7. Summary and outlook

Based on refs. [14], [17], [18] and [21], the off-shell $\mathcal{N} = (1, 0)$ and on-shell $\mathcal{N} = (1, 1)$ harmonic superfield approaches were accounted for. It was argued that they are very efficient for constructing higher-dimensional invariants in the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories. The novel solution of the $\mathcal{N} = (1, 1)$ SYM constraints in terms of harmonic $\mathcal{N} = (1, 0)$ superfields was given. This allowed us to explicitly construct the full set of the dimensions $\underline{d} = 8$ and $\underline{d} = 10$ superfield invariants revealing $\mathcal{N} = (1, 1)$ on-shell supersymmetry.

All possible $\underline{d} = 6$ $\mathcal{N} = (1, 1)$ invariants were shown to be on-shell vanishing, thus proving the UV finiteness of $\mathcal{N} = (1, 1)$ SYM at one loop.

The off-shell $\underline{d} = 8$ invariants which would be non-vanishing on shell, are absent. Assuming that the $\mathcal{N} = (1, 0)$ supergraphs yield integrals over the full $\mathcal{N} = (1, 0)$ harmonic superspace, this means the absence of two-loop counterterms as well.

Two dimension $\underline{d} = 10$ invariants were constructed as integrals over the whole $\mathcal{N} = (1, 0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over $\mathcal{N} = (1, 1)$ superspace, while the double-trace one cannot. This property combined with an additional reasoning (e.g., based on the algebraic renormalization scheme [25]) could explain why the double-trace invariant is UV protected.

Some further lines of development:

(a). It would be tempting and instructive to construct the $\underline{d} \geq 12$ invariants in the $\mathcal{N} = (1, 1)$ SYM theory using the on-shell $\mathcal{N} = (1, 1)$ harmonic superspace techniques and to see whether they exhibit the properties similar to the $\underline{d} = 10$ invariants. It is probable that the proper corrections to the hidden supersymmetry transformations will be of need, while tackling this issue.

(b). It is worth to apply the same method for constructing the Born-Infeld action with the manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries.

(c). The closely related problem is to recover the higher dimension invariants listed above from the quantum $\mathcal{N} = (1, 0)$ supergraph techniques. The first steps in this direction were undertaken in a recent paper [30].

(d). An interesting task is to develop an analogous on-shell bi-harmonic $\mathcal{N} = 4, 4D$ superspace approach to the $\mathcal{N} = 4, 4D$ SYM theory in the $\mathcal{N} = 2$ superfield formulation (by solving the $\mathcal{N} = 4$ SYM constraints in terms of $\mathcal{N} = 2$ superfields) and apply it to the problem of constructing the $\mathcal{N} = 4$ SYM effective action. It is curious that such a formulation has not been constructed so far in full generality, despite the existence of various more sophisticated superspace formulations (see [31] for a recent review).

(e). As was mentioned in sect. 5, the crucial last step in solving the constraints of $\mathcal{N} = (1, 1)$ SYM theory was identifying the vector connections in the sectors with the standard and “hatted” harmonics. Only after this identification, the constraints in the bi-harmonic superspace get fully equivalent to those in the standard setting and imply the equations of motion for the involved superfields. It would be interesting to develop a superspace formulation with two *independent* vector connections and to inquire whether it could give rise to an off-shell description of $\mathcal{N} = (1, 1), 6D$ (and, perhaps, of $\mathcal{N} = 4, 4D$) SYM theories. The introduction of two vector connections seems to imply doubling of the x -coordinate, in an obvious parallel with the recent eight-dimensional reformulation of $\mathcal{N} = 4, 4D$ SYM theory in [32].

(f). Applications in supergravity? Lacking the double-trace divergent structures in the 3-loop amplitude in $\mathcal{N} = (1, 1)$ SYM theory is similar to the absence of 3-loop and 4-loop divergences for the four-graviton amplitudes in $\mathcal{N} = 4, 4D$ and $\mathcal{N} = 5, 4D$ supergravities [26], [27], [28], [29]. All these UV divergence cancelations could find a common explanation within the harmonic superspace approach ¹.

Acknowledgments

I thank the organizers of the conference ISQS’2016 for the kind hospitality in Prague. I am grateful to my co-authors Guillaume Bossard, Andrei Smilga and Boris Zupnik. A partial support from the RFBR grant no. 15-02-06670, grant of Russian Science Foundation no. 16-12-10306 and a grant of Heisenberg-Landau program is acknowledged.

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¹ For a recent relevant discussion see [33].

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